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2000 J. Phys. A: Math. Gen. 33 4081

(http://iopscience.iop.org/0305-4470/33/22/311)

J. Phys. A: Math. Gen. **33** (2000) 4081–4094. Printed in the UK PII: S0305-4470(00)08365-7

Supersymmetric extension of the integrable box–ball system

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Received 1 October 1999, in final form 7 March 2000

Abstract. We introduce the supersymmetric extension of the box–ball system. By use of the isomorphism of the crystal base for the super Lie algebra, we define the time evolution operator, and give the evolution equation explicitly. We also construct the soliton solutions.

1. Introduction

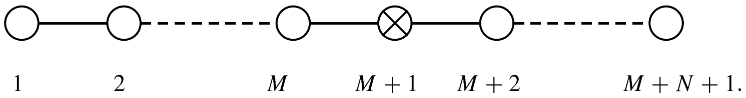
Cellular automata have been widely studied in various fields of physics [1]. Among them, there exists an *integrable* family of cellular automata. The most well known one is that introduced in [2], called the box–ball system (BBS). This BBS has a kind of soliton solution, and one may expect that a connection with the soliton equation exists. Indeed, this problem was solved [3] by introducing a method of 'ultra-discretization', and it was established that the BBS in [2] is a certain limit of the KdV-type integrable difference–difference equation [4, 5]. We can then define the BBS associated with the $\mathfrak{sl}(M)$ Lie algebra from a discrete analogue of the KP equation (Hirota–Miwa equation) [6].

Recently we have found another important interpretation of the BBS [7]. The evolution of the BBS has been regarded as the configuration at zero temperature ('crystallization') of the integrable vertex model on a two-dimensional square lattice with appropriate boundary conditions. After that work, the *crystallization* method was re-formulated by use of the crystal base [8–10], and a new type of BBS was introduced in [9], based on the crystal base for arbitrary classical Lie algebra. Unfortunately, the evolution rule thereof is very involved, and it is difficult to obtain an explicit form of the evolution equation, and to see a relationship with a difference analogue of the soliton equation. In this paper, alternatively we use the supersymmetric extension ($\mathbb{Z}_2$-grading) of the Lie algebra, and define a new BBS. The notion of the supersymmetry becomes clear when we regard the auxiliary space as a 'carrier' [11]. We can give evolution equations explicitly, and we propose two-soliton solutions from a combinatorics viewpoint.

This paper is organized as follows. In section 2 we give a brief review on the crystal base for the super Lie algebra. The crystal isomorphism is explicitly written following [12]. In section 3 we define the evolution operator based on the crystal base. The 'soliton' is introduced as the semi-standard tableau. In section 4 we explicitly introduce an evolution equation for the supersymmetric extension of the BBS. We clarify a meaning of the *supersymmetry* from a BBS viewpoint. We further construct the soliton solutions explicitly. We conjecture that the combinatorial property appears not only in the soliton scattering but in the S-matrix in the τ-functions. The final section is devoted to concluding remarks.

2. Crystal base for superalgebra

We briefly review the crystal base for the superalgebra $U_q(\mathfrak{sl}(M+1|N+1))$ following [12]. Here we use slightly different notation from that in [12], and the Dynkin diagram [13] is drawn as follows:



As was stressed in [12], the representation for the super Lie algebra is not completely reducible in general, and here we use a certain class of representation stable under tensor products. It will become clear that the property of the tensor product is essential in constructing the BBS [7]. We set the crystal base $\boldsymbol{B}$ for the fundamental representation as

$$\boldsymbol{B} = \boldsymbol{B}_+ \sqcup \boldsymbol{B}_-$$

where

$$\boldsymbol{B}_+ = \{0, 1, 2, \ldots, M\}$$
$$\boldsymbol{B}_- = \{M+1, M+2, \ldots, M+N, M+N+1\}.$$

The action of the Kashiwara operator [14] is summarized in the crystal graph

$$\boxed{0} \xrightarrow{1} \boxed{1} \xrightarrow{2} \cdots \xrightarrow{M} \boxed{M} \xrightarrow{M+1} \boxed{M+1} \longrightarrow \cdots \xrightarrow{M+N+1} \boxed{M+N+1}. \tag{2.1}$$

Here, as usual, $b \xrightarrow{i} b'$ denotes $\tilde{f}_i b = b'$. For this vector space of the fundamental module, we assign an ordering on $\boldsymbol{B}$ by

$$0 < 1 < 2 < \cdots < M < M+1 < \cdots < M+N+1.$$

We set $b \in B$ as an element of the crystal B, and define $\varepsilon_i(b)$ and $\varphi_i(b)$ for $i \in \{1, 2, \ldots, M+N+1\}$ respectively, as

$$\varepsilon_i(b) = \max\{n \in \mathbb{Z}_{\geqslant 0} | \tilde{e}_i{}^n b \neq 0\}$$
$$\varphi_i(b) = \max\{n \in \mathbb{Z}_{\geqslant 0} | \tilde{f}_i{}^n b \neq 0\}.$$

For crystals B_1 and B_2, we can define the tensor product $B_1 \otimes B_2 = \{b_1 \otimes b_2 | b_1 \in B_1, b_2 \in B_2\}$, and the Kashiwara operators act as follows:

- $i = 1, \ldots, M$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if} \quad \varphi_i(b_1) \geqslant \varepsilon_i(b_2) \\ b_1 \otimes \tilde{e}_i(b_2) & \text{if} \quad \varphi_i(b_1) < \varepsilon_i(b_2) \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if} \quad \varphi_i(b_1) > \varepsilon_i(b_2) \\ b_1 \otimes \tilde{f}_i(b_2) & \text{if} \quad \varphi_i(b_1) \leqslant \varepsilon_i(b_2) \end{cases}$$

- $i = M+1$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} \tilde{e}_i(b_1) \otimes b_2 & \text{if} \quad \varphi_i(b_1) + \varepsilon_i(b_1) = 1 \\ \sigma b_1 \otimes \tilde{e}_i(b_2) & \text{if} \quad \varphi_i(b_1) + \varepsilon_i(b_1) = 0 \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} \tilde{f}_i(b_1) \otimes b_2 & \text{if} \quad \varphi_i(b_1) + \varepsilon_i(b_1) = 1 \\ \sigma b_1 \otimes \tilde{f}_i(b_2) & \text{if} \quad \varphi_i(b_1) + \varepsilon_i(b_1) = 0 \end{cases}$$

- $i = M+2, \ldots, M+N+1$

$$\tilde{e}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{e}_i(b_2) & \text{if} \quad \varphi_i(b_2) \geqslant \varepsilon_i(b_1) \\ \tilde{e}_i(b_1) \otimes b_2 & \text{if} \quad \varphi_i(b_2) < \varepsilon_i(b_1) \end{cases}$$

$$\tilde{f}_i(b_1 \otimes b_2) = \begin{cases} b_1 \otimes \tilde{f}_i(b_2) & \text{if} \quad \varphi_i(b_2) > \varepsilon_i(b_1) \\ \tilde{f}_i(b_1) \otimes b_2 & \text{if} \quad \varphi_i(b_2) \leqslant \varepsilon_i(b_1). \end{cases}$$

Here σ is the parity operator,

$$\sigma(\tilde{e}_i) = (-)^{p(i)} \tilde{e}_i \qquad \sigma(\tilde{f}_i) = (-)^{p(i)} \tilde{f}_i$$

with $p(i) = 1$ for $i = M+1$, and $p(i) = 0$ otherwise.

To parametrize the tensor product, we use the semi-standard tableau (see, e.g., [15]). Here the 'semi-standard' tableau is defined as a tableau obtained from a Young diagram by filling the boxes with elements of $\boldsymbol{B}$ subject to the following two constraints [12]:

- the entries in each row are increasing, allowing the repetition of elements in $\boldsymbol{B}_+$, but not permitting the repetition of elements in $\boldsymbol{B}_-$ and
- the entries in each column are increasing, allowing the repetition of elements in $\boldsymbol{B}_-$, but not permitting the repetition of elements in $\boldsymbol{B}_+$.

It was proved [12] that, for any $(M+1, N+1)$-hook Young diagram Y, we can embed the crystal structure, and that the crystal $B(Y)$ is connected. Note that the Young tableau and the Kac–Dynkin diagram for the super Lie algebra were also discussed in [16].

Hereafter, we use $B_\ell = B\Bigg(\underbrace{\boxed{}\boxed{}\boxed{\cdots}\boxed{}}_{\ell}\Bigg)$ for brevity, and we consider the crystal isomorphism, $B_\ell \otimes B_m \stackrel{\sim}{\to} B_m \otimes B_\ell$. In a simple case, we have the crystal isomorphism,

$$B_\ell \otimes \boldsymbol{B} \stackrel{\sim}{\longrightarrow} \boldsymbol{B} \otimes B_\ell$$

as follows [12]:

- If $\boxed{b}\boxed{a_1}$ is semi-standard, then

$$\boxed{a_1}\boxed{a_2}\boxed{\cdots}\boxed{a_\ell} \otimes \boxed{b} \stackrel{\sim}{\longmapsto} \boxed{a_\ell} \otimes \boxed{b}\boxed{a_1}\boxed{\cdots}\boxed{a_{\ell-1}}$$

- If $\begin{array}{|c|}\hline a_1 \\ \hline b \\ \hline\end{array}$ is semi-standard, then

$$\boxed{a_1}\boxed{\cdots}\boxed{a_j}\boxed{\cdots}\boxed{a_\ell} \otimes \boxed{b} \stackrel{\sim}{\longmapsto} \boxed{a_j} \otimes \boxed{a_1}\boxed{\cdots}\boxed{b}\boxed{\cdots}\boxed{a_\ell}$$

where j is the largest integer such that $\begin{array}{|c|}\hline a_j \\ \hline b \\ \hline\end{array}$ is semi-standard.

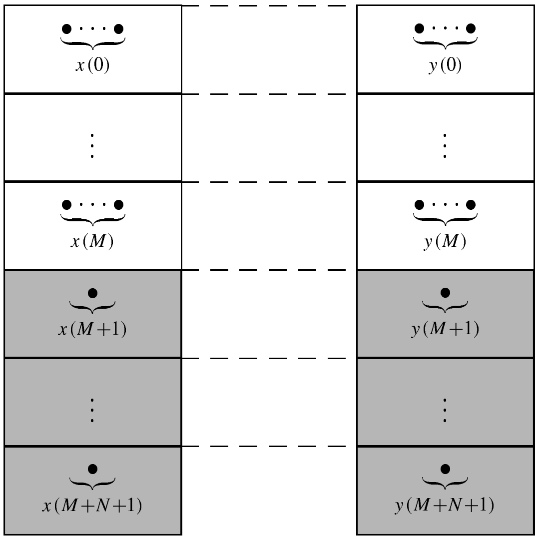
Based on a fact that the 'admissible reading'

$$\begin{array}{ccc} B_\ell & \longrightarrow & \boldsymbol{B}^{\otimes \ell} \\ \boxed{a_1}\boxed{\cdots}\boxed{a_\ell} & \longmapsto & \boxed{a_\ell} \otimes \cdots \otimes \boxed{a_1} \end{array}$$

induces a crystal structure on B_ℓ [12], we have the following rule as an extension of [17] to obtain the crystal isomorphism,

$$\begin{aligned} B_\ell \otimes B_m &\stackrel{\sim}{\longrightarrow} B_m \otimes B_\ell \\ b_1 \otimes b_2 &\stackrel{\sim}{\longmapsto} b_2' \otimes b_1'. \end{aligned}$$

Here we suppose $\ell \geqslant m$. We set $b_1 = \boxed{a_1 \mid a_2 \mid \cdots \mid a_\ell}$, $b_2 = \boxed{a'_1 \mid a'_2 \mid \cdots \mid a'_m}$ with $x(i) = \#\{k | a_k = i\}$, $y(i) = \#\{k | a'_k = i\}$. We represent $b_1 \otimes b_2$ by the two column diagrams. Each column has $M + N + 2$ rows, and we put $x(i)$ (resp. $y(i)$) dots $\bullet$ in the ith row of the left (resp. right) column. Note that $0 \leqslant x(i), y(i) \leqslant 1$ if $i \in \boldsymbol{B}_-$. We call the ith box the bosonic (resp. fermionic) box when $i \in \boldsymbol{B}_+$ (resp. $i \in \boldsymbol{B}_-$). The shaded box denotes the fermionic one. The role of the dotted lines will be clarified later.

$b_1 \otimes b_2 =$ 

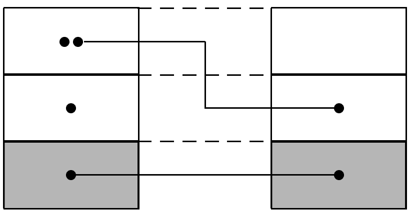
(2.2)

(i) Pick the dot $\bullet_a$ which is located in the highest position in the right column. If the dot $\bullet_a$ is in the bosonic box, connect it with the dot in the left column which has the lowest position among all dots whose positions are higher than that of $\bullet_a$. When there is no such dot on the left, return to the bottom. If the dot $\bullet_a$ is in the fermionic box, the partner in the left column is the dot which is the lowest among all dots whose positions are higher than or equal to that of $\bullet_a$.

(ii) Repeat the previous process for the remaining unconnected dots in the right column.

(iii) The crystal isomorphism, $b_1 \otimes b_2 \overset{\sim}{\mapsto} b'_2 \otimes b'_1$, is obtained by sliding the remaining unpaired dots in the left column to the right one.

An example below which is the $\mathfrak{sl}(2,1)$ case indicates the crystal isomorphism, $\boxed{0 \mid 0 \mid 1 \mid 2} \otimes \boxed{1 \mid 2} \overset{\sim}{\longmapsto} \boxed{0 \mid 2} \otimes \boxed{0 \mid 1 \mid 1 \mid 2}$.



3. Evolution operator and soliton

We shall introduce the evolution operator of the BBS by the crystal isomorphism, and consider a notion of a 'soliton'.

3.1. Evolution operator

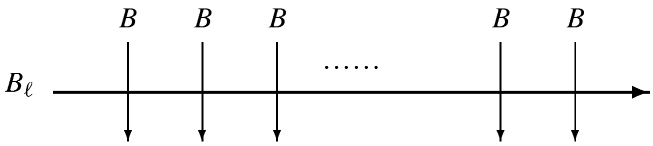
The idea which was introduced in [7] and was developed in [8–10] is to define the evolution operator T_ℓ by

$$T_\ell : \boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_L} \longmapsto \boxed{a'_1} \otimes \boxed{a'_2} \otimes \cdots \otimes \boxed{a'_L} \tag{3.1}$$

where we have the crystal isomorphism,

$$\begin{aligned} &\underbrace{\boxed{0 \mid 0 \mid \cdots \mid 0}}_{\ell} \otimes \boxed{a_1} \otimes \boxed{a_2} \otimes \cdots \otimes \boxed{a_L} \\ &\qquad \overset{\sim}{\longmapsto} \boxed{a'_1} \otimes \boxed{a'_2} \otimes \cdots \otimes \boxed{a'_L} \otimes \underbrace{\boxed{0 \mid 0 \mid \cdots \mid 0}}_{\ell}. \end{aligned} \tag{3.2}$$

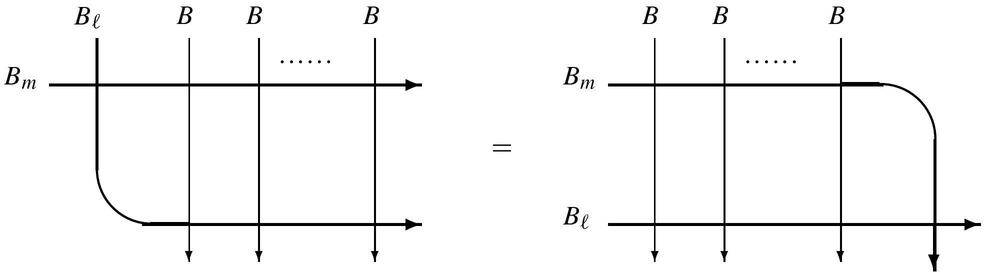
Here we set $L \gg 1$, and $a_j = a'_k = 0$ for sufficiently large j, k. This can be drawn schematically as



By definition of the crystal base, the evolution operators commute with each other:

$$[T_\ell, T_m] = 0 \tag{3.3}$$

where ℓ and m are arbitrary positive integers. Note that this relation is depicted as



Hereafter we use

$$T = \lim_{\ell \to \infty} T_\ell \tag{3.4}$$

and we consider the BBS associated with this evolution operator T.

3.2. Soliton and scattering

We call the following configuration a 'soliton' of length m:

$$\cdots \otimes \boxed{0} \otimes \underbrace{\boxed{a_1} \otimes \cdots \otimes \boxed{a_m}}_{m} \otimes \boxed{0} \otimes \boxed{0} \otimes \cdots \tag{3.5}$$

where $a_j \neq 0$ and $\boxed{a_m \mid \cdots \mid a_2 \mid a_1}$ is semi-standard. We can see from the crystal isomorphism rule that the soliton (3.5) propagates rightward by step $\min(\ell, m)$ when the configuration is evolved by the operator T_ℓ (3.1). In the following we denote the above soliton as $[a_1 a_2 \ldots a_m]$ for brevity.

Following a strategy presented in [10] we can see that the scattering of solitons is factorized into two-body scattering based on the commutativity of the evolution operator (3.3). It is important that the *velocity* of the soliton is suppressed when we use the evolution operator T_ℓ which commutes with T, and that we can change the order of the scattering to obtain the out-going state.

Furthermore, as in the case of the $\mathfrak{sl}(M+1)$ BBS, the two-soliton scattering in our BBS,

$$[a_1 \cdots a_\ell] \times [b_1 \cdots b_m] \to [b'_1 \cdots b'_m] \times [a'_1 \cdots a'_\ell]$$

coincides with the crystal isomorphism for $\mathfrak{sl}(M|N+1)$:

$$\begin{array}{ccc} B_\ell \otimes B_m & \stackrel{\sim}{\longrightarrow} & B_m \otimes B_\ell \\ \boxed{a_\ell \,|\, \cdots \,|\, a_1} \otimes \boxed{b_m \,|\, \cdots \,|\, b_1} & \stackrel{\sim}{\longmapsto} & \boxed{b'_m \,|\, \cdots \,|\, b'_1} \otimes \boxed{a'_\ell \,|\, \cdots \,|\, a'_1} \end{array}$$

where we should shift the crystal base (2.1) for $\mathfrak{sl}(M|N+1)$ as $\boxed{1} \to \cdots \to \boxed{M+N+1}$. This is proved, owing to the fact that $U_q(\mathfrak{sl}(M|N+1))$ is a subalgebra of $U_q(\mathfrak{sl}(M+1|N+1))$, by checking for the 'genuine' lowest weight [12].

4. Evolution equation

We now introduce the evolution equations of the BBS constructed by the evolution operator T (3.4), and give the soliton solutions explicitly. To relate the crystal base $\boldsymbol{B}$ to the BBS, we interpret the crystal base $b \in \boldsymbol{B}$ as

$$\boxed{0} : \text{empty box}$$

$$\boxed{a} : \text{box is occupied by ball } a \ (a \neq 0).$$

Due to a supersymmetry we classify ball a by

$$\text{ball } a : \begin{cases} \text{bosonic} & \text{for} \quad 1 \leqslant a \leqslant M \\ \text{fermionic} & \text{for} \quad M+1 \leqslant a \leqslant M+N+1. \end{cases}$$

Throughout this section, $u^t_{n,a}$ denotes the number of balls a in the nth box at time t. Variables $v^t_{n,a}$ denote the number of balls in the auxiliary space (or the carrier), and satisfy a condition [7]

$$u^t_{n,j} + v^t_{n,j} = u^{t+1}_{n,j} + v^t_{n+1,j} \tag{4.1}$$

for $j = 1, 2, \ldots, M+N+1$.

4.1. $\mathfrak{sl}(2|1)$

The first nontrivial example is for $\mathfrak{sl}(2|1)$, because the $\mathfrak{sl}(1|1)$ case corresponds to the BBS with the carrier whose capacity is one [8]. We find that the evolution equation of the BBS defined by the evolution operator T (3.4) is explicitly written as

$$u^{t+1}_{n,1} = \min[v^t_{n,1}, 1 - u^t_{n,1} - u^{t+1}_{n,2}] \tag{4.2a}$$

$$u^{t+1}_{n,2} = \min[v^t_{n,2}, 1 - u^t_{n,1}]. \tag{4.2b}$$

Note that we have a constraint (4.1) for $j = 1, 2$, and that $u^t_{n,1} = u^t_{n,2} = 0$ means that the nth box is empty at time t.

The evolution equation (4.2) can be interpreted in a picture of the BBS:

(i) We take the leftmost ball 2 (fermionic) out of its box, and put it in the first empty box to its right.

(ii) We take the new leftmost ball 2 as long as it has not yet been moved at this time step and not been overtaken by the previously moved ball 2. We then move it to the first empty box to its right. We leave the overtaken ball 2 as it is.
(iii) We continue the previous process until every ball 2 has been moved. Note that the overtaken ball 2 can be regarded as having moved.
(iv) We take the leftmost ball 1, and move it to the first empty box to its right.
(v) Take the new leftmost unmoved ball 1, and move it to the first empty box to its right.
(vi) Continue the previous process till every ball 1 has been moved. These steps (i)–(vi) represent a unit time step.
(vii) Repeat the above processes.

A rule for moving balls depends on whether the ball is fermionic or bosonic.

From a definition of the semi-standard tableau, the auxiliary 'carrier' has one ball 2 at most, and we can easily see the *fermionic* property of ball 2. See that the rule for ball 1, which is *bosonic*, is the same as an original BBS in [2].

We give several examples of the evolutions below.

(a) $[211] \times [1] \to [2] \times [111]$

$$\begin{aligned} t = 0 &: 021100010000000000 \\ 1 &: 000021101000000000 \\ 2 &: 000000020111000000 \\ 3 &: 000000002000111000 \end{aligned}$$

(b) $[211] \times [2] \to [2] \times [211]$

$$\begin{aligned} t = 0 &: 021100020000000000 \\ 1 &: 000021102000000000 \\ 2 &: 000000021210000000 \\ 3 &: 000000000202110000 \\ 4 &: 000000000020002110 \end{aligned}$$

(c) $([111] \times [21]) \times [2] \to [1] \times [21] \times [211]$

$$\begin{aligned} t = 0 &: 011102100020000000000000000000 \\ 1 &: 000011021102000000000000000000 \\ 2 &: 000000110021210000000000000000 \\ 3 &: 000000001100202110000000000000 \\ 4 &: 000000000011020002110000000000 \\ 5 &: 000000000000112000002110000000 \\ 6 &: 000000000000001210000002110000 \\ 7 &: 000000000000000102100000002110 \end{aligned}$$

(d) $[111] \times ([21] \times [2]) \to [1] \times [21] \times [211]$

$$\begin{aligned} t = 0 &: 01110000210200000000000000 \\ 1 &: 00001110002120000000000000 \\ 2 &: 00000001110022100000000000 \\ 3 &: 00000000001112021000000000 \\ 4 &: 00000000000001210211000000 \\ 5 &: 00000000000000010210021100. \end{aligned}$$

The last two examples show that the out-going state does not depend on the order of collisions, as explained at the end of section 3.1.

One sees that the evolution is very similar to the BBS associated with $\mathfrak{sl}(3)$ [7]. We should note again that a difference is that the carrier can only have at most one ball 2 while an infinite number of balls 1 can occupy the carrier simultaneously.

We consider the soliton solution. We set the dynamical variables by use of the ultradiscrete τ-function as [6, 8, 18]

$$u^t_{n,1} = Y^t_n + X^t_{n+1} - Y^t_{n+1} - X^t_n \qquad u^t_{n,2} = Y^{t-1}_{n+1} + X^t_n - Y^{t-1}_n - X^t_{n+1}. \quad (4.3)$$

Substituting the above equation into the evolution equation (4.2), we obtain the following equations:

$$X^t_n + Y^{t+1}_{n+1} = \max[Y^t_n + X^{t+1}_{n+1}, X^t_{n+1} + Y^{t+1}_n - 1] \quad (4.4a)$$

$$X^{t+1}_{n+1} + X^t_n + Y^{t-1}_n = \max[2X^t_n + Y^t_{n+1}, X^{t+1}_n + X^t_{n+1} + Y^{t-1}_n - 1]. \quad (4.4b)$$

In contrast to the $\mathfrak{sl}(M)$ case [8], we obtain both the bilinear and trilinear equations. Based on the numerical experiments of the soliton solutions of the $\mathfrak{sl}(M)$ BBS [8, 18], we have obtained some of the soliton solutions as follows:

(1) *One-soliton solution.*

$$Y^t_n = \max[0, C + n - pt - p]$$
$$X^t_n = \max[0, C + n - pt - x]$$

where C is arbitrary integer, and x denotes the number of '2' in the soliton. Because ball 2 is fermionic, we have $x \in \{0, 1\}$. The length of soliton $p \geqslant x$ denotes a velocity of the soliton. It is remarked that the one-soliton solution for the $\mathfrak{sl}(3)$ case has the same form but x is arbitrary.

(2) *Two-soliton solution.*
We set p_1 and p_2 as velocities of solitons ($p_r \in \mathbb{Z}_+$), and suppose that $p_1 \geqslant p_2$. We define the τ-functions as

$$Y^t_n = \max[0, C_1 + n - p_1(t+1), C_2 + n - p_2(t+1),$$
$$C_1 + C_2 + 2n - (p_1 + p_2)(t+1) - S_1]$$
$$X^t_n = \max[0, C_1 + n - p_1 t - x, C_2 + n - p_2 t - x,$$
$$C_1 + C_2 + 2n - (p_1 + p_2)t - 2x - S_2]$$

where C_r is arbitrary, and the scattering matrix is given by

$$S_1 = 2p_2 - y \qquad S_2 = 2p_2 - x$$

with $x, y \in \{0, 1\}$. This solution describes the following scattering of the two solitons:

x	y	:	$\overbrace{\qquad}^{p_1} \times \overbrace{\qquad}^{p_2} \to \overbrace{\qquad}^{p_2} \times \overbrace{\qquad}^{p_1}$
0	0	:	$[11\ldots1] \times [11\ldots1] \to [11\ldots1] \times [11\ldots1]$
0	1	:	$[11\ldots1] \times [21\ldots1] \to [11\ldots1] \times [21\ldots1]$
1	0	:	$[21\ldots1] \times [11\ldots1] \to [21\ldots1] \times [11\ldots1]$
1	1	:	$[21\ldots1] \times [21\ldots1] \to [21\ldots1] \times [21\ldots1]$

One can check that the soliton scattering obeys the crystal isomorphism, $B_{p_1} \otimes B_{p_2} \xrightarrow{\sim} B_{p_2} \otimes B_{p_1}$, for $\mathfrak{sl}(1|1)$.

4.2. $\mathfrak{sl}(1|2)$

We have two kinds of ball as in the case of $\mathfrak{sl}(2|1)$, but here each ball acts like a *fermion*. We find from the crystal isomorphism in section 2 that the evolution equation for the evolution operator T is written by

$$u_{n,1}^{t+1} = \min[v_{n,1}^t, 1 - u_{n,2}^{t+1}]$$
$$u_{n,2}^{t+1} = \min[v_{n,2}^t, 1 - u_{n,1}^t]$$

where, as before, we have a constraint (4.1). This evolution equation can also be interpreted by the BBS picture, and the rule to move each ball as the BBS is similar to the rule in section 4.2; we only replace (v) with (v′),

(v′) Take the leftmost ball 1, which is unmoved and is not overtaken by the previously moved ball 1, and move it to the first empty box to its right. We leave the ball 1 which was overtaken as it is.

As seen from the definition of the semi-standard tableau, we have only three kinds of soliton: [2], [1] and [21]. The velocity of the first two is unity, while the last one has twice the velocity. We substitute equations (4.3) into the evolution equations (4.5), and we obtain two trilinear equations,

$$X_n^t + Y_n^t + Y_{n+1}^{t+1} = \max[X_{n+1}^{t+1} + 2Y_n^t, X_n^t + Y_{n+1}^t + Y_n^{t+1} - 1] \qquad (4.6a)$$
$$X_{n+1}^{t+1} + X_n^t + Y_n^{t-1} = \max[2\,X_n^t + Y_{n+1}^t, X_n^{t+1} + X_{n+1}^t + Y_n^{t-1} - 1]. \qquad (4.6b)$$

Unfortunately, we do not know the general soliton solutions and we explicitly give some soliton solutions below:

(1) *One-soliton solution.*

$$Y_n^t = \max[0, C + n - pt - p]$$
$$X_n^t = \max[0, C + n - pt - x]$$

where C is arbitrary, and we have $(p, x) = (2, 1), (1, 0), (1, 1)$.

(2) *Two-soliton solution.*

$$Y_n^t = \max[0, C_1 + n - 2(t+1), C_2 + n - (t+1), C_1 + C_2 + 2n - 3(t+1) - 1]$$
$$X_n^t = \max[0, C_1 + n - 2t - 1, C_2 + n - t - x, C_1 + C_2 + 2n - 3t - x - 2]$$

where C_j is arbitrary, and $x = \{0, 1\}$. We give two typical examples below:

(a) $x = 0$, i.e., $[21] \times [1] \to [1] \times [21]$

$t = 0$: 02100100000000
1 : 00021010000000
2 : 00000211000000
3 : 00000001210000
4 : 00000000102100

(b) $x = 1$, i.e., $[21] \times [2] \to [2] \times [21]$

$t =$ 0 : 02102000000000
1 : 00021200000000
2 : 00000221000000
3 : 00000020210000
4 : 00000002002100

One also sees that the soliton scattering, [21] × [1] and [21] × [2], obeys the crystal isomorphism, $B_2 \otimes B_1 \overset{\sim}{\to} B_1 \otimes B_2$ for $\mathfrak{sl}(0|2)$.

4.3. $\mathfrak{sl}(M+1|N+1)$

We shall derive the evolution equations for the BBS associated with the supersymmetric $\mathfrak{sl}(M+1|N+1)$ algebra. For our convention, we use the notation

$$\boldsymbol{B}'_+ = \boldsymbol{B}_+ \setminus \{0\} = \{1, 2, \ldots, M\} \qquad \boldsymbol{B}_- = \{M+1, M+2, \ldots, M+N+1\}.$$

From the definitions of the semi-standard tableau and the evolution operators, we can check that the evolution equations can be summarized as follows:

$$u_{n,j}^{t+1} = \min\left[v_{n,j}^t, 1 - \sum_{i=1}^{j} u_{n,i}^t - \sum_{i=j+1}^{M+N+1} u_{n,i}^{t+1}\right] \qquad \text{for} \quad j \in \boldsymbol{B}'_+ \tag{4.7a}$$

$$u_{n,j}^{t+1} = \min\left[v_{n,j}^t, 1 - \sum_{i=1}^{j-1} u_{n,i}^t - \sum_{i=j+1}^{M+N+1} u_{n,i}^{t+1}\right] \qquad \text{for} \quad j \in \boldsymbol{B}_- \tag{4.7b}$$

where

$$u_{n,j}^t + v_{n,j}^t = u_{n,j}^{t+1} + v_{n+1,j}^t \qquad \text{for} \quad j = 1, 2, \ldots, M+N+1. \tag{4.8}$$

The interpretation of the BBS is simple. We have M and $N+1$ kinds of bosonic and fermionic ball respectively, and a rule of the BBS is as follows:

(i) Set $a = M + N + 1$.
(ii) Take the leftmost ball a out of its box, and put it in the first empty box to its right.
(iii) When the ball a is bosonic, take the leftmost ball a as long as it has not been moved at this time step. If the ball a is fermionic, take the leftmost ball a as long as it has not yet been moved and further it has not been overtaken by the moved ball a. Then put it in the first empty box to its right.
(iv) Continue the process (iii) until every ball a has been moved. Note that, when ball a is fermionic, the overtaken ball a is regarded as having been moved.
(v) Continue the same processes (ii)–(iv) for other balls, $a = M + N, M + N - 1, \ldots, 2, 1$. Steps (i)–(v) represent one time step.
(vi) Repeat the processes (i)–(v).

To construct the soliton solutions of the BBS, we set the dynamical variables in terms of the ultra-discretized τ-functions as

$$\begin{aligned} u_{n,j}^t &= Y(t,n,j) + Y(t,n+1,j+1) - Y(t,n+1,j) - Y(t,n,j+1) \\ v_{n,j}^t &= Y(t+1,n,j) + Y(t,n,j+1) - Y(t,n,j) - Y(t+1,n,j+1) \end{aligned} \tag{4.9}$$

where we suppose a condition,

$$Y(t,n,1) = Y(t+1,n,M+N+2).$$

By substituting (4.9) into the evolution equations (4.7), we obtain the bilinear and trilinear equations as follows:

$$\begin{aligned} Y(t+1,n+1,j) + Y(t,n,j+1) = \max[&Y(t,n,j) + Y(t+1,n+1,j+1), \\ &Y(t,n+1,j+1) + Y(t+1,n,j) - 1] \qquad \text{for} \quad j \in \boldsymbol{B}'_+ \end{aligned} \tag{4.10a}$$

$$\begin{aligned} Y(t,n,j) + Y(t+1,n+1,j) + Y(t,n,j+1) = \max[&2Y(t,n,j) + Y(t+1,n+1,j+1), \\ &Y(t+1,n,j) + Y(t,n+1,j) + Y(t,n,j+1) - 1] \qquad \text{for} \quad j \in \boldsymbol{B}_-. \end{aligned} \tag{4.10b}$$

A set of the bilinear equations is same as that for the $\mathfrak{sl}(M+1)$ BBS [8, 18], while the trilinear equation is new. Below, we give the one- and two-soliton solutions.

(1) *One-soliton solution.*
When the soliton $[a_1 \dots a_p]$ propagates, we can set the τ-function as

$$Y(t,n,j) = \max\left[0, C + n - pt - \sum_{i=j}^{M+N+1} x(i)\right] \tag{4.11}$$

where C is an arbitrary integer, and p is the length of the soliton. Variables $x(i)$ denote the number of i in the soliton, $x(i) = \#\{k|a_k = i\}$.

(2) *Two-soliton solution.*
We consider the scattering of two solitons,

$$[a_1 \dots a_{p_1}] \times [b_1 \dots b_{p_2}] \to [b'_1 \dots b'_{p_2}] \times [a'_1 \dots a'_{p_1}]$$

where we suppose $p_1 \geqslant p_2$. To write the soliton solutions, we use the *column* interpretation (2.2) of the crystal base. Note that, in the soliton scattering in the $\mathfrak{sl}(M+1|N+1)$ case, we use the crystal base for $\mathfrak{sl}(M|N+1)$, and shift the bases as $\boxed{1} \to \cdots \to \boxed{M+N+1}$. We conjecture that the τ-function is given by

$$\begin{aligned} &Y(t,n,j) = \max[0, \xi_1(n,t,j), \xi_2(n,t,j), \xi_1(n,t,j) + \xi_2(n,t,j) - S(j)] \\ &\xi_r(n,t,j) = C_r + n - p_r t - \sum_{i=j}^{M+N+1} x_r(i) \qquad \text{for} \quad r = 1, 2. \end{aligned} \tag{4.12}$$

Here C_r is arbitrary, and we set

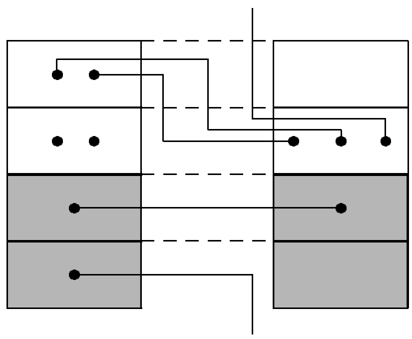
$$x_1(i) = \#\{k|a_k = i\} \qquad x_2(i) = \#\{k|b'_k = i\}.$$

Note that $x_2(i)$ depends on the out-going soliton state. The scattering matrix $S(j)$ is

$$S(j) = p_2 + H(j) \tag{4.13}$$

where $H(j)$ is the number of lines that cross the jth dotted line between two columns when we represent two solitons by the diagram (2.2) and apply a rule for the isomorphism.

For example, we consider [432211] × [3222] scattering for the $\mathfrak{sl}(3|2)$ case (balls 3 and 4 are fermionic). The out-going state is determined as [4311] × [322222], and the scattering parameters $H(j)$ are given by $H(1) = 1$, $H(2) = 3$, $H(3) = 0$, $H(4) = 0$:



The solution (4.12) gives all the two-soliton solutions. This form has not been known even in the $\mathfrak{sl}(M)$ case.

We list scattering data for the $\mathfrak{sl}(1|3)$ case below, where all three balls are fermionic, and the scattering matrix, $S(j) = p_2 + H(j)$, is determined by the above rule.

in-coming $\to$ out-going	:	$H(1)$	$H(2)$	$H(3)$
$[32] \times [3] \to [3] \times [32]$	:	0	0	0
$[32] \times [2] \to [2] \times [32]$	:	0	0	0
$[32] \times [1] \to [3] \times [21]$	:	1	0	0
$[31] \times [3] \to [3] \times [31]$	:	0	0	0
$[31] \times [2] \to [1] \times [32]$	:	0	1	0
$[31] \times [1] \to [1] \times [31]$	:	0	0	0
$[21] \times [3] \to [2] \times [31]$	:	0	0	1
$[21] \times [2] \to [2] \times [21]$	:	0	0	0
$[21] \times [1] \to [1] \times [21]$	:	0	0	0
$[321] \times [3] \to [3] \times [321]$	:	0	0	0
$[321] \times [2] \to [2] \times [321]$	:	0	0	0
$[321] \times [1] \to [1] \times [321]$	:	0	0	0
$[321] \times [32] \to [32] \times [321]$	:	0	0	0
$[321] \times [31] \to [31] \times [321]$	:	0	0	0
$[321] \times [21] \to [21] \times [321]$	:	0	0	0

See that the scattering rule is governed by the crystal isomorphism for $\mathfrak{sl}(0, 3)$.

We note that both the out-going states and the scattering matrix depend on the underlying Lie algebra even if the in-coming state is the same. See the following examples:

algebra	:	in-coming $\to$ out-going	:	$H(1)$	$H(2)$	$H(3)$	$H(4)$
$\mathfrak{sl}(5)$, $\mathfrak{sl}(4, 1)$	:	$[432111] \times [321] \to [421] \times [332111]$	:	1	1	1	0
$\mathfrak{sl}(3, 2)$	:	$[432111] \times [321] \to [431] \times [322111]$	:	1	1	0	0
$\mathfrak{sl}(2, 3)$	:	$[432111] \times [321] \to [432] \times [321111]$	:	1	0	0	0

5. Concluding remarks

We have introduced a new integrable BBS system based on the super Lie algebra. We have used the crystal isomorphism (3.1) to define the evolution operator. We have further given the explicit form of the evolution equation (4.7), and constructed the soliton solutions by use of the τ-function. It is interesting that our two-soliton solution (4.12) has a combinatorial property not only in the scattering state but also in the S-matrix in the τ-function. We remark that we obtain essentially the same evolution equation as (4.7) when we use another tensor product,

$$\begin{array}{|c|}\hline M+N+1 \\ \hline \vdots \\ \hline M+N+1 \\ \hline\end{array} \otimes \boxed{a_1} \otimes \cdots \otimes \boxed{a_L} \overset{\sim}{\longmapsto} \boxed{a_1'} \otimes \cdots \otimes \boxed{a_L'} \otimes \begin{array}{|c|}\hline M+N+1 \\ \hline \vdots \\ \hline M+N+1 \\ \hline\end{array}$$

In this case, the two-soliton scattering can be described by the crystal isomorphism $B(1^\ell) \otimes B(1^m) \xrightarrow{\sim} B(1^m) \otimes B(1^\ell)$, where

$$B(1^\ell) = B\left(\left.\begin{array}{|c|}\hline \\ \hline \vdots \\ \hline \\ \hline\end{array}\right\}\ell\right).$$

The evolution equation (4.7) can be related to the ultra-discrete analogue of the Toda-type equation as was discussed in [6] for the $\mathfrak{sl}(M)$ case. We suppose that our BBS associated with $\mathfrak{sl}(M+1|N+1)$ has K solitons. We set $Q_n^{t(i)}$ (for $i \in \boldsymbol{B}'_+ \sqcup \boldsymbol{B}_-$) as the number of balls i in the nth soliton from the left at time t. We further introduce $E_n^{t(M+N+1)}$ as the number of empty boxes between the nth and the $(n+1)$th solitons at time t. Based on the rule of our supersymmetric BBS, the evolution equations are given by

$$Q_n^{t+1(i)} = \min\left[\sum_{j=1}^{n} Q_j^{t(i)} - \sum_{j=1}^{n-1} Q_j^{t+1(i)}, E_n^{t(i)}\right] \qquad \text{for} \quad i \in \boldsymbol{B}'_+ \tag{5.1a}$$

$$Q_n^{t+1(i)} = \min\left[\sum_{j=1}^{n} Q_j^{t(i)} - \sum_{j=1}^{n-1} Q_j^{t+1(i)}, E_n^{t(i)} + Q_{n+1}^{t(i)}\right] \qquad \text{for} \quad i \in \boldsymbol{B}_- \tag{5.1b}$$

$$E_n^{t(i-1)} = Q_{n+1}^{t(i)} + E_n^{t(i)} - Q_n^{t+1(i)}. \tag{5.1c}$$

Each variable $E_n^{t(i)}$ (for $i = 1, 2, \ldots, M+N$) is determined by (5.1c) recursively, and it denotes the number of empty boxes between the nth and $(n+1)$th solitons just before $Q_n^{t+1(i)}$ is fixed by (5.1a) or (5.1b). We note that we set $E_0^{t(i)} = E_K^{t(i)} = \infty$, and suppose a periodicity, $E_n^{t(0)} = E_n^{t+1(M+N+1)}$. When the dynamical variables $E_n^{t(j)}$ (for $n = 0, 1, \ldots, K$) and $Q_n^{t(j)}$ (for $n = 1, 2, \ldots, K$) are respectively written as

$$E_n^{t(i)} = X_{n+1}^{t(i)} + X_{n-1}^{t+1(i)} - X_n^{t(i)} - X_n^{t+1(i)} \tag{5.2a}$$

$$Q_n^{t(i)} = X_{n-1}^{t(i)} + X_n^{t(i-1)} - X_n^{t(i)} - X_{n-1}^{t(i-1)} \tag{5.2b}$$

we find that the ultradiscrete τ-function satisfies a set of the bilinear and the trilinear equations as (4.10)

$$\begin{aligned} &X_n^{t+1(i)} + X_n^{t(i+1)} \\ &\quad = \min[X_n^{t(i)} + X_n^{t+1(i+1)}, X_{n+1}^{t(i+1)} + X_{n-1}^{t+1(i)}] \qquad \text{for} \quad i \in \boldsymbol{B}'_+ \end{aligned} \tag{5.3a}$$

$$\begin{aligned} &X_n^{t+1(i)} + X_n^{t(i+1)} + X_n^{t(i)} \\ &\quad = \min[2X_n^{t(i)} + X_n^{t+1(i+1)}, X_n^{t(i)} + X_n^{t(i+1)} + X_{n-1}^{t+1(i)}] \qquad \text{for} \quad i \in \boldsymbol{B}_- \end{aligned} \tag{5.3b}$$

where $n = 1, 2, \ldots, K$, and we have $X_{-1}^{t(i)} = X_{K+1}^{t(i)} = \infty$ and $X_0^{t(i)} = 0$.

In [3], a relationship is established between Takahashi's BBS and the integrable difference–difference equation called the Hirota equation [4]. The procedure to derive the BBS from the difference–difference equation is called ultra-discretization. When we *simply* apply an inverse of ultra-discretization to our new BBS (4.10), we obtain a set of the following bilinear and trilinear equations:

$$\begin{aligned} &(1+\delta)\tau(t+1, n+1, j)\tau(t, n, j+1) \\ &\qquad = \tau(t, n, j)\tau(t+1, n+1, j+1) + \delta\tau(t, n+1, j+1)\tau(t+1, n, j) \\ &\qquad \text{for} \quad j \in \boldsymbol{B}'_+ \end{aligned} \tag{5.4a}$$

$$\begin{aligned} &(1+\delta)\frac{\tau(t+1, n+1, j)}{\tau(t+1, n, j)} = \frac{\tau(t, n, j)\tau(t+1, n+1, j+1)}{\tau(t+1, n, j)\tau(t, n, j+1)} + \delta\frac{\tau(t, n+1, j)}{\tau(t, n, j)} \\ &\qquad \text{for} \quad j \in \boldsymbol{B}_-. \end{aligned} \tag{5.4b}$$

It is natural to expect that our equations are related to the supersymmetric KP hierarchy, which has been widely studied in [19–25]. Unfortunately, the difference–difference equation for the supersymmetric KP hierarchy is not known, and it is unclear whether a ultra-discretization does work for the supersymmetric KP hierarchy which includes the Grassmannian variables.

In closing we comment on the energy function. We have used the supersymmetric crystal base to construct the integrable BBS. It was pointed out that the phase shift for the soliton scattering in the original BBS is determined by the energy function (H-function, which is $H(1)$ in our definition) for the affine Lie algebra [10], which can be expected from our result that the S-matrix in the τ-function is given by the function $H(j)$. Study on the phase shift and the function $H(j)$ in our BBS will be useful to compute the character formula for the affine super Lie algebra by use of a method of the path configuration sum, and to see a relationship with a result in [26].

Acknowledgments

The authors would like to thank T Tokihiro for stimulating discussions. Thanks are also to G Hatayama, A Kuniba and T Takagi for useful communications. One of the authors (RI) is a Research Fellow of the Japanese Society for the Promotion of Science.

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